18.821 Project 2: Happy Sequences

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Abstract

For a doubly infinite sequence S of 0's and 1's, we consider the operation $\mathscr{F}$ which produces another such sequence $\mathscr{F}(S)$ from S, defined by the assignment $0 \mapsto 10, 1 \mapsto 100$. A sequence is *happy* if it is in the image of $\mathscr{F}^m$ for all m. We construct a happy sequence H_0, and prove that it is the unique happy sequence having an origin. We also compute the density of 0's and 1's in H_0, and prove that an arbitrary happy sequence is quasiperiodic.

Contents

1 Introduction

In this paper, we will explore a particular kind of infinite sequences of 1s and 0s called *happy sequences*. Consider the following operation, which we will call $\mathscr{F}$, on a finite or infinite sequence of 0's and 1's: replace each 1 by 100 and each 0 by 10. We say a sequence is *lucky* if it can be obtained by this procedure from another sequence, and we say it is *very very ... very lucky* (where the word *very* appears $m-1$ times), or *m-times lucky*, if it can be obtained by applying this operation m times to another sequence. A *happy sequence* is a sequence that is infinite in both directions that is m-times lucky for any integer value m. To simplify notation for the rest of the paper, we will let a denote 1 and b denote 0. For example, 10010 will be written as ab^2ab.

It is not immediately obvious that any happy sequences exist. We thus begin our study by proving the following theorem.

Theorem 1.1. There exists a happy sequence H_0.

We prove Theorem 1.1 in section 2 of the paper, where we use an inductive process that involves applying the operator $\mathscr{F}$ infinitely many times to a starting string, ba, to generate a sequence we will then prove is happy. This method produces a specific happy sequence, and it is unclear whether there is another method that can produce a different happy sequence. This leads us to wonder if happy sequences are unique, or in other words if this happy sequence H_0 we constructed is the only happy sequence. At the end of section 2, we conjecture, under the strong assumption that happy sequences have a fixed origin, that happy sequences are unique.

Knowing happy sequences exist, we begin to study some of their properties. Section 3 is centered around the periodicity of arbitrary happy sequences. We prove the following theorem.

Theorem 1.2. Happy sequences are not periodic. However, happy sequences are *quasi*periodic, in the sense that every substring in a happy sequence appears infinitely many times.

After proving each substring repeats infinitely many times, we explore the pattern in which a's and b's appear in these substrings and in the sequence as a whole by studying the density of a's and b's in happy sequences. In section 4, we use two specific properties of lucky sequences to bound the density of a's and b's on a general happy sequence.

Theorem 1.3. The density ρ_a of a's in a happy sequence is bounded by $\frac{1}{3} \leq \rho_a \leq \frac{1}{2}$ and the density ρ_b of b's in a happy sequence is bounded by $\frac{1}{2} \leq \rho_b \leq \frac{2}{3}$.

Then, we solve a system of equations based on the growth pattern of a's and b's under the operation $\mathscr{F}$ to explicitly find the density of a's and b's in the happy sequence H_0 we constructed.

Theorem 1.4. The density of a's in our happy sequence H_0, $\rho_a(H_0)$, is $\sqrt{2}-1$. The density of b's, $\rho_b(H_0)$, is $2-\sqrt{2}$.

Last, we examine how happy sequences change under a modified operator $\mathscr{F}_m$, as the original choice for $\mathscr{F}$ to map a to ab^2 and b to ab is arbitrary. In section 5, we define this operator $\mathscr{F}_m$ to replace a's with $abb\ldots b$ (a followed by m b's) and b's with $ab\ldots b$ (a followed by $m-1$ b's). We define $m-$happy sequences in the same way we defined happy sequences in section 2 but using the modified $\mathscr{F}_m$. Then using the methods from section 2 and section 4, we prove $m-$happy sequences exist for all m and find the density of a's and b's for the specific $m-$happy sequences we found.

2 Properties and existence of happy sequences

In order to prove results about lucky and happy sequences, we will first define them rigorously in this section. We next prove some properties of lucky sequences that will be useful throughout the paper. In Theorem 2.1, we answer the basic question of existence of happy sequences in the positive. We will construct a sequence H_0 such that $\mathscr{F}(H_0) = H_0$, that is, a sequence that is fixed by $\mathscr{F}$. It will follow that H_0 is a happy sequence. Finally, we prove in Theorem 2.2 that H_0 is the unique happy sequence having an origin.

2.1 Operation $\mathscr{F}$, lucky sequences, and happy sequences

Let us define lucky and happy sequences, as well as our operations $\mathscr{F}$ and $\mathscr{G}$. Let a *sequence* be an ordered collection of elements of $\{a, b\}$ indexed by $\mathbb{Z}$. We denote it as $\{c_n\}_{n\in\mathbb{Z}}$ or $\ldots c_{-1}c_0c_1\ldots$. Let $\mathcal{S}$ be the set of sequences. We will also use finite analogues of sequences for our paper, which we call *strings.* A *string* is a tuple $(c_1, c_2, \ldots, c_n)$ with entries in $\{a, b\}$. We will also denote this tuple as $c_1c_2\ldots c_n$. We say that n is the *length* of the string, and define the length function ℓ by $\ell(c_1c_2\ldots c_n) = n$. Let $\mathcal{F}$ be the set of strings.

We define the operation that we will be applying to strings and sequences. Let $\mathscr{F} : \{a, b\} \to \{ab^2, ab\}$ be an operation on the letters a given by $\mathscr{F}(a) = ab^2$ and $\mathscr{F}(b) = ab$. We can extend $\mathscr{F}$ to strings and sequences as $\mathscr{F} : \mathcal{F} \to \mathcal{F}$ and $\mathscr{F} : \mathcal{S} \to \mathcal{S}$ by $\mathscr{F}(c_1c_2\ldots c_n) = \mathscr{F}(c_1)\mathscr{F}(c_2)\ldots\mathscr{F}(c_n)$ and $\mathscr{F}(S) = \ldots\mathscr{F}(s_{-1})\mathscr{F}(s_0)\mathscr{F}(s_1)\ldots$, respectively. We also denote $\mathscr{F}^n := \underbrace{\mathscr{F}\circ\cdots\circ\mathscr{F}}_{n\text{ times}}$.

We can now define our sequences of interest. We say that a sequence S is *lucky* if $S = \mathscr{F}(S')$ for some $S' \in \mathcal{S}$. A sequence S is *very lucky* if $S = \mathscr{F}^2(S')$ for some $S' \in \mathcal{S}$. Generalizing, a sequence S is very very ... very lucky, where $m - 1$ is the number of times the word *very* shows up, if $S = \mathscr{F}^m(S')$ for some $S' \in \mathcal{S}$. We will also call these sequences *m-times lucky sequence* for convenience. Let $\mathcal{L}^m$ be the set of m-times lucky sequences. The following proposition characterizes all lucky sequences:

Proposition 2.1. A lucky sequence is any sequence of only ab's and ab^2's.

Proof. A lucky sequence must be produced by $\mathscr{F}$ applied to a sequence of b's and a's. Each b becomes a ab and each a becomes a ab^2, so the resulting sequence (a lucky one) must be a sequence of ab's and ab^2's. □

For lucky sequences, it makes sense to talk about the inverse of $\mathscr{F}$. Let $\mathscr{G} : \{ab^2, ab\} \to \{a, b\}$ be an operation given by $\mathscr{G}(ab^2) = a$ and $\mathscr{G}(ab) = b$. We can extend $\mathscr{G}$ to lucky sequences as $\mathscr{G} : \mathcal{L}^1 \to \mathcal{S}$ by $\mathscr{G}(s) = \ldots\mathscr{G}(s_{-1})\mathscr{G}(s_0)\mathscr{G}(s_1)\ldots$, where $s_i \in \{ab^2, ab\}$. We can always express a lucky sequence s in this way by Proposition 2.1. We define $\mathscr{G}^m : \mathcal{L}^m \to \mathcal{S}$ by $\mathscr{G}^n := \underbrace{\mathscr{G}\circ\cdots\circ\mathscr{G}}_{n\text{ times}}$.

A *happy sequence* is a sequence S such that, for every m, there exists a sequence S_m such that $S = \mathscr{F}^m(S_m)$. Alternatively, S is happy if $\mathscr{G}^m(S)$ is well-defined for every m. Let $\mathcal{H} = \mathcal{L}^\infty$ be the set of happy sequences. It is not clear how happy sequences should look like. Furthermore, it is not even clear that happy sequences exist. In fact, showing that they exist and exploring their properties are going to be the main goals of this paper. The first tiny step that we will make in this direction will be proving the following requirement for a sequence to be happy, which we will repeatedly use.

Proposition 2.2. If H is a happy sequence, then H cannot contain the strings a^2 or b^3.

Proof. As H is lucky, by Proposition 2.1, H is a sequence of only ab's and ab^2's. When we concatenate any two of these, we get the following strings:

$$abab, abab^2, ab^2ab, ab^2ab^2.$$

In particular, we do not get any strings a^3 and b^2, as desired. $\square$

2.2 Existence of happy sequences

Let us construct a doubly infinite sequence of a's and b's explicitly and then show it is happy. This will suffice to show the happy sequences exist. The following lemma will be useful for our construction.

Lemma 2.1. We have $\mathscr{F}^{n-1}(ba) \subset \mathscr{F}^n(ba)$ for $n \geq 1$, where $\mathscr{F}^0(ba) := ba$ by convention.

Proof. We proceed by induction. For $n = 1$, $ba \subset ab\underline{ba}b = ab^2ab = \mathscr{F}(ba)$, so the result holds. Assume the result holds for n. That is, assume that $\mathscr{F}^n(ba) = c\mathscr{F}^{n-1}(ba)d$ for some strings c, d. For $n + 1$, we would have that

$$\mathscr{F}^n(ba) \subset \mathscr{F}(c)\mathscr{F}^n(ba)\mathscr{F}(d) = \mathscr{F}(c\mathscr{F}^{n-1}(ba)d) = \mathscr{F}(\mathscr{F}^n(ba)) = \mathscr{F}^{n+1}(ba)$$

so our induction, and thus our proof, is complete. $\square$

Lemma 2.2. If we apply $\mathscr{F}$ successively to the string ba, we get a sequence H_0.

Proof. By Lemma 2.1, applying $\mathscr{F}$ successively is equivalent to concatenating strings to ba to the left and the right:

$$\begin{aligned} ba &= ba \\ \mathscr{F}(ab) &= abab^2 \\ \mathscr{F}^2(ab) &= ab^2abab^2abab \\ &\vdots \end{aligned}$$

Thus, we define H_0 to be the sequence obtaining by concatenating all of these strings. That is, H_0 would look as follows:

$$H_0 = \ldots ab^2abab^2abab\ldots.$$

$\square$

Lemma 2.3. We have $\mathscr{F}^m(H_0) = H_0$ for all $m \geq 1$.

Proof. By induction, it suffices to show that $\mathscr{F}(H_0) = H_0$. Our strategy will be to keep track of a *center* $|$, i.e, a position that will allow us to index our happy sequence as $\ldots, -2, -1, 0, 1, 2, \ldots$. We will keep track of the center by noting that applying $\mathscr{F}$ does not change our previous choice for $|$. The process is as follows:

$$\begin{aligned}
ba &= b|a \\
\mathscr{F}(ab) &= ab|abb \\
\mathscr{F}^2(ab) &= abbab|abbabab \\
&\vdots \\
H_0 &= \ldots abbab|abbabab\ldots.
\end{aligned}$$

Because of Lemma 2.1, $\mathscr{F}^{n-1}(ba) \subset \mathscr{F}^n(ba)$, so in particular, the same center that we chose for $\mathscr{F}^{n-1}(ba)$ also works for $\mathscr{F}^n(ba)$. That is, the center $|$ does not change when we apply $\mathscr{F}$. With this center as a reference, if we enumerate letters to the left as $0, -1, -2, \ldots$ and letters to the right as $1, 2, 3, \ldots$, it is clear that the n-th letter of H_0 is the same letter as the n-th letter of $\mathscr{F}(H_0)$. Hence, $\mathscr{F}(H_0) = H_0$, as desired. $\square$

Theorem 2.1. There exists at least one happy sequence; in particular, H_0 is a happy sequence.

Proof. It suffices to show that H_0 is happy. However, by Lemma 2.3, given any m, we have that

$$H_0 = \mathscr{F}^m(H_0)$$

so H_0 is in the image of $\mathscr{F}^m$ for all m, and thus is happy, as desired. $\square$

2.3 Uniqueness of happy sequences assuming there is an origin

A general goal is to classify happy sequences up to some form of equivalence. Since finite shifts do not fundamentally change a sequence infinite in both directions, we adopt the convention that two sequences $\{c_n\}_{n\in\mathbb{Z}}$ and $\{c'_n\}_{n\in\mathbb{Z}}$ are *equivalent* if there exists an integer k such that $c_n = c'_{n+k}$ for all $n \in \mathbb{Z}$. We conjecture that all happy sequences are equivalent to H_0. Even though we have not been able to verify this in general, we do so for a particular class of happy sequences, which we call *happy sequences with an origin*.

We say that a sequence is *infinite to the right* if it is of the form $c_0c_1c_2\dots$, and we say that it is *infinite to the left* it has the form $\dots c_{-2}c_{-1}c_0$. In the same way as we defined it for happy sequences, we say that a sequence is *right happy* if it is infinite to the right and it is happy. Similarly, we define *left happy* sequences. A happy sequence *has an origin* if it consists of a left happy concatenated with a right happy sequence to the right. The following uniqueness result holds.

Lemma 2.4. *The sequences $\mathscr{F}^\infty(a)$ and $\mathscr{F}^\infty(b)$ are the only right and left happy sequences respectively.*

Proof. Suppose $R = c_0c_1c_2\dots$ is a right happy sequence. Because it is a sequence formed from the strings ab^2, ab, we have $c_0 = a$. Moreover, the sequence $\mathscr{G}^m(R)$ is happy for all m, so its first entry is a as well. Thus $R = \mathscr{F}^m(\mathscr{G}^m(R))$ has the form $\mathscr{F}^m(a)c_k\dots$ for all $m \geq 0$. Hence we must have $R = \mathscr{F}^\infty(a)$.

An analogous proof works for a left happy sequence $L = c_{-2}c_{-1}c_0$. We must have $c_0 = b$, and hence L has the form $\dots c_{-k}\mathscr{F}^m(b)$ for all $m \geq 0$. $\square$

Lemma 2.4 has the following immediate consequence.

Theorem 2.2. *If a happy sequence H has an origin, then $H = H_0$.*

Proof. By Lemma 2.4, $H = \mathscr{F}^\infty(b)\mathscr{F}^\infty(a) = H_0$. $\square$

3 Quasiperiodicity of happy sequences

A simple way to build sequences that are infinite to the left and right is to build periodic sequences. These are sequences which are formed by repeating some string infinitely many times. For example, the sequence

$$\dots bababababa\dots$$

is obtained by repeating the string ba. It is a natural question to ask if periodic sequences can be happy, as this would give an easy way to construct happy sequences. In this section, we will show in Proposition 3.1 that no happy sequence is periodic. After introducing the idea of *blocks*, we show in Theorem 3.1 that happy sequences satisfy the weaker property of being *quasiperiodic*. Before, we briefly introduce the notion of *substrings*.

A *substring* f of length k in a sequence $S = \{c_n\}_{n\in\mathbb{Z}}$ is a string of the form $c_nc_{n+1}\dots c_{n+k-1}$ with $n \in \mathbb{Z}, k \in \mathbb{Z}_{>0}$. We denote a substring by the notation $f \subseteq S$, and write $\ell(f) = k$ to indicate that f has length k. For example, the string ba is a substring of H_0, so we have $ba \subset H_0$ with $\ell(ba) = 2$. The same definition applies when S is not a doubly infinite sequence, and is infinite in only one direction or a string. In this case we use identical notation.

Proposition 3.1. A happy sequence is not periodic.

Proof. Suppose $H = \{c_n\}_{n\in\mathbb{Z}}$ is a happy sequence which is periodic with period N. Then $H = \mathscr{F}(H')$, where H' is also a happy sequence. We claim H' is periodic with period strictly less than N. Without loss of generality, we can assume that $c_0 = a$. Then $c_N = a$, and H is obtained by repeating the string $c_0c_1\ldots c_{N-1}$. We claim the string $c_0c_1\ldots c_{N-1}$ is formed from the strings ab^2, ab. Since H is happy, it is itself formed from the strings ab^2, ab by Proposition 2.1, so it is sufficient to show that $c_{N-1} = b$. Indeed, if $c_{N-1} = a$, then H contains the substring $c_{N-1}c_N = aa$, which is not possible, since H is a sequence formed from the strings ab^2, ab.

It follows that is some string s of length strictly less than N with $\mathscr{F}(s) = c_0c_1\ldots c_{N_1}$, hence S' is periodic with period s. Iterating this process, there is some happy periodic constant sequence S_0 such that $\mathscr{F}^N(S_0) = S$. Constant sequences are not happy, as they are not sequences formed from the strings ab^2, ab, so we have reached a contradiction. □

Although happy sequences are not periodic, they do satisfy the weaker property of being quasiperiodic.

Definition 3.1. A sequence S is *quasiperiodic* if any substring in the sequence appears infinitely many times.

We show that happy sequences are quasiperiodic using the idea of *blocks*. A *block* is a finite sequence obtained by applying $\mathscr{F}$ to the strings a, b a finite number of times. For example, the strings $\mathscr{F}^2(b) = ab^2ab$ and $\mathscr{F}^1(a) = ab^2$ are blocks. We will first prove that every substring is contained in a block, and will then show that any block in a happy sequence occurs infinitely many times.

Lemma 3.1. Any substring of a happy sequence is contained in a block.

Proof. We first show that any length 2 substring of a happy sequence H is contained in a block. Indeed, since H is formed from the strings ab^2, ab by Proposition 2.1, the only length 2 substrings in H are ab, b^2, ba. These substrings are all contained in the block $\mathscr{F}^2(a) = ab^2abab$.

Let $s \subset H$ be a substring of a happy sequence H. We claim that there exists a substring $s' \subset \mathscr{G}(H)$ with $\ell(s') \leq 2$ or $\ell(s') < \ell(s)$ such that $s \subseteq \mathscr{F}(s')$. For this, we have four cases.

1. Assume s is of the form $s = a\ldots b$. In this case, s consists of the concatenation of strings ab or ab^2, so we can choose $s' = \mathscr{G}(s)$, and as $\mathscr{G}$ maps $ab \mapsto b$ and $ab^2 \mapsto a$,
$$\ell(s') \leq \frac{\ell(s)}{2} < \ell(s).$$

2. Assume s is of the form $s = a \dots a$. As H is happy, the letter directly to the right of s would be a b, so we can now consider the substring $sb = a \dots ab$, which consists of the concatenation of strings ab or ab^2. Thus, if we choose $s' = \mathscr{G}(sa)$,
$$\ell(s') \leq \frac{\ell(sa)}{2} = \frac{\ell(s) + 1}{2} \leq \ell(s).$$
Here, equality holds if and only if $\ell(s) = 1 \leq 2$, as required.

3. Assume s is of the form $s = b \dots b$. As H is happy, there must be an a or ab string directly to the left of s, so we can now consider the substring $cs = cb \dots b$, where $c = ab$ or a. Thus, if we choose $s' = \mathscr{G}(cs)$,
$$\ell(s') \leq \frac{\ell(cs)}{2} \leq \frac{\ell(s) + 2}{2} \leq \ell(s).$$
Here, equality holds if and only if $\ell(s) = 2$, as required.

4. Assume s is of the form $s = b \dots a$. If $\ell(s) \leq 2$, we are done. Consider the case $\ell(s) = 3$. In this case, s can only be $s = b^2 a$. In particular, there must be an a to the left of s and a b to its right, so by choosing $s' = \mathscr{G}(asb) = \mathscr{G}(ab^2ab) = ab$, we get that $\ell(s') = 2 < 3 = \ell(s)$, as desired.

   Now, assume that $\ell(s) > 3$. As H is happy, there must be an a or ab string directly to the left of s, and there must be a b directly to the right of s, so we can now consider the substring $csb = cb \dots ab$, where $c = ab$ or a. Let t be s without the first letter b. Thus, if we choose $s' = \mathscr{G}(csb)$,
$$\ell(s') \leq \frac{\ell(csb)}{2} = \frac{\ell(cb)}{2} + \frac{\ell(tb)}{2} \leq \frac{3}{2} + \frac{\ell(s)}{2} < \ell(s),$$
as required.

Thus, by analyzing all the cases, there is always a substring $s' \subset \mathscr{G}(H)$ with $\ell(s') \leq 2$ or $\ell(s') < \ell(s)$ such that $s \subseteq \mathscr{F}(s')$. Now, $\mathscr{G}(H)$ is also happy, so if $\ell(s') > 2$, we can apply the same argument to $s' \subset \mathscr{G}(H)$ to get some s'' with $\ell(s'') \leq 2$ or $\ell(s'') < \ell(s') < \ell(s)$ that satisfies $s \subseteq \mathscr{F}^2(s'')$. By repeating this process, as the length is a strictly decreasing sequence of integers when it is greater than 2, we will get to some $\tilde{s}$ with $\ell(\tilde{s}) \leq 2$ and $s \subseteq \mathscr{F}^m(\tilde{s})$. Since $\tilde{s}$ is contained in a block by the first claim in the proof, the string s is as well. □

Theorem 3.1. A happy sequence is quasiperiodic.

Proof. We claim any block appears an infinite number of times in a happy sequence. Suppose H is a happy sequence and $B \subset H$ is a block. Assume that $B = \mathscr{F}^N(s)$, where s is one of the strings a, b. Since H is happy, there is some sequence S with $\mathscr{F}^{N+1}(S) = H$. By definition of the operator $\mathscr{F}$, the strings a, b appear infinitely many times in $\mathscr{F}(S)$, hence B occurs infinitely many times in H.

Consider now any string $s \subseteq H$. By Lemma 4.2, s is contained in some block B. Furthermore, as we just showed, B occurs infinitely many times in H. Hence, $s \subseteq B$ also occurs infinitely many times in H, so H is quasiperiodic, as desired. $\square$

4 Density of a's and b's in a happy sequence

Now that we have established happy sequences are quasiperiodic, meaning every substring appears an infinite number of times, it is natural to ask about the frequency in which each value appears in these subtrings and in the sequence as a whole. This leads us to explore a key quality of happy sequences, the density of a's and b's. In this section, we establish bounds on the densities of a's and b's in happy sequences. We use Proposition 2.2 to find these bounds based on the number of a's and the number of b's can appear consecutively in a happy sequence.

Definition 4.1. The density ρ_a of a's in a happy sequence is the limit as n approaches infinity of the number of a's, $N_a(-n, n)$, in the string from index $-n$ to index n around an arbitrary origin divided by the total number of elements in that string, $2n + 1$. Mathematically this can be represented as $\rho_a = \lim_{n\to\infty} \frac{N_a(-n,n)}{2n+1}$. A happy sequence consists only of a's and b's, so $\rho_a + \rho_b = 1$ where ρ_b is the density of b's in a happy sequence.

Proposition 4.1. The density ρ_a of a's in a happy sequence is bounded by $\frac{1}{3} \le \rho_a \le \frac{1}{2}$ and the density ρ_b of b's in a happy sequence is bounded by $\frac{1}{2} \le \rho_b \le \frac{2}{3}$.

Proof. Proposition 2.2 states that a happy sequence cannot have two a's in a row or three b's in a row. This implies that we must have at most one a in each substring of length 2 and at least one a in each substring of length 3. We thus have a lower bound of $\frac{1}{3}$ and an upper bound of $\frac{1}{2}$ on the density ρ_a of a's. The density ρ_b of b's follows immediately as the density of a's and b's must sum to 1. $\square$

The bounds on the density of a's and b's apply to all happy sequences, including ones that may exist but have not been explicitly constructed like H_0 was. Without having explicitly found other happy sequences, we cannot find exact densities of their a's and b's. Since H_0 is constructed in an concrete way, though, we can find the exact density of a's and b's of this particular sequence.

Theorem 4.1. The density of a's in our happy sequence H_0, $\rho_a(H_0)$, is $\sqrt{2}-1$. The density of b's, $\rho_b(H_0)$, is $2-\sqrt{2}$.

Proof. Let a_n and b_n be the number of a's and b's of $\mathscr{F}^n(ba)$. Note that, as $a \mapsto ab^2$ and $b \mapsto ab$, we have that

$$a_{n+1} = a_n + b_n \text{ and } b_{n+1} = 2a_n + b_n.$$

Writing these identities in matrix form, we obtain the matrix equation

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}. \tag{1}$$

We can write (1) in terms of the initial number of a's and b's, a_0 and b_0, and apply our recurrence matrix n times to get the numbers of a's and b's after any amount of applications of $\mathscr{F}$. The equation is as follows:

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^n \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}.$$

The first step in solving for a_n and b_n is to diagonalize the recurrence matrix. The diagonalization is as follows:

$$\begin{aligned} \begin{pmatrix} a_n \\ b_n \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{pmatrix}^n \begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2\sqrt{2}} \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} (1+\sqrt{2})^n & 0 \\ 0 & (1-\sqrt{2})^n \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2\sqrt{2}} \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}. \end{aligned} \tag{2}$$

Next, we multiply the matrices in (2). We get the following:

$$\begin{aligned} \begin{pmatrix} a_n \\ b_n \end{pmatrix} &= \begin{pmatrix} \frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2} & \frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}} \\ \frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{\sqrt{2}} & \frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2} \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2} a_0 + \frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}} b_0 \\ \frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{\sqrt{2}} a_0 + \frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2} b_0 \end{pmatrix}. \end{aligned} \tag{3}$$

We can use the values of a_n and b_n in (3), which are in terms of n, to find the ratio of a's to the total number of a's and b's as n approaches infinity. The result will be the density of a's in the happy sequence. In computing the limit, we use the idea that a value between 0 and 1 approaches 0 when it is raised to the nth power as n approaches infinity. In mathematical terms, $\lim_{n\to\infty} \varepsilon^n = 0$

for all $\varepsilon \in \mathbb{R}$ with $0 < |\varepsilon| < 1$. The limit is calculated as follows:

$$\begin{aligned}
\rho_a(H_0) &= \lim_{n\to\infty} \frac{a_n}{a_n + b_n} \\
&= \lim_{n\to\infty} \frac{\frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2}a_0 + \frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}b_0}{\frac{(1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}}{2}a_0 + \frac{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}}{2\sqrt{2}}b_0} \\
&= \lim_{n\to\infty} \frac{\frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2}a_0 + \frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}b_0}{\frac{(1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}}{2}a_0 + \frac{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}}{2\sqrt{2}}b_0} \cdot \frac{\frac{1}{(1+\sqrt{2})^{n+1}}}{\frac{1}{(1+\sqrt{2})^{n+1}}} \\
&= \lim_{n\to\infty} \frac{\frac{1+\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^n}{2(1+\sqrt{2})}a_0 + \frac{1-\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^n}{2\sqrt{2}(1+\sqrt{2})}b_0}{\frac{1+\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{n+1}}{2}a_0 + \frac{1-\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{n+1}}{2\sqrt{2}}b_0} \\
&= \frac{\frac{1}{1+\sqrt{2}} \cdot \left(\frac{1}{2}a_0 + \frac{1}{2\sqrt{2}}b_0\right)}{\frac{1}{2}a_0 + \frac{1}{2\sqrt{2}}b_0} \\
&= \frac{1}{1+\sqrt{2}} \\
&= \sqrt{2} - 1.
\end{aligned}$$

From the definition of density we immediately have that $\rho_b(H_0) = 1-(\sqrt{2}-1) = 2-\sqrt{2}$. □

Remark. Note that the density does not depend on the initial number of a's and b's. In particular, this proof works for $F^\infty(a)$ and $F^\infty(b)$ as well.

5 Modifying the operator $\mathscr{F}$

In this section, we modify the operator $\mathscr{F}$ to be a function of some integer $m \geq 2$ so that $\mathscr{F}_m : \{a, b\} \to \{ab^m, ab^{m-1}\}$. We define a new type of happy sequence here, call it an m-happy sequence. An m-happy sequence is defined in the same way a happy sequence is but the operator applied is $\mathscr{F}_m$ instead of $\mathscr{F}$. We will prove m-happy sequences exist and then use the same matrix multiplication approach as in section 4 to find the exact density of a's and b's in this sequence. Let us begin by explicitly constructing one such m-happy sequence to prove that these sequences exist. Our argument is the same as in section 2.

Lemma 5.1. We have $\mathscr{F}_m^{n-1}(ba) \subset \mathscr{F}_m^n(ba)$ for $n \geq 1$, where $\mathscr{F}_m^0(ba) := ba$ by convention.

Proof. We proceed by induction. For $n = 1$, $ba \subset ab^{m-1}\underline{ba}b^{m-1} = ab^m ab^{m-1} = \mathscr{F}(ba)$, so the result holds. Assume the result holds for n. That is, assume that

$\mathscr{F}_m^n(ba) = c\mathscr{F}_m^{n-1}(ba)d$ for some strings c, d. For $n+1$, we would have that

$$\mathscr{F}_m^n(ba) \subset \mathscr{F}_m(c)\mathscr{F}_m^n(ba)\mathscr{F}_m(d) = \mathscr{F}_m(c\mathscr{F}_m^{n-1}(ba)d) = \mathscr{F}_m^{n+1}(ba)$$

so our induction, and thus our proof, is complete. $\square$

Lemma 5.2. If we apply $\mathscr{F}_m$ successively to the string ba, we get a sequence $H_{0,m}$.

Proof. By Lemma 5.1, applying $\mathscr{F}_m$ successively is equivalent to concatenating strings to ba to the left and the right:

$$\begin{aligned}
ba &= ba \\
\mathscr{F}_m(ab) &= ab^{m-1}ab^m \\
&= a(b\dots b)a(bb\dots b) \\
\mathscr{F}_m^2(ab) &= ab^m(ab^{m-1})^{m-1}ab^m(ab^{m-1})^m \\
&= ab^m(ab^{m-1}\dots ab^{m-1})ab^m(ab^{m-1}ab^{m-1}\dots ab^{m-1}) \\
&\vdots \\
H_{0,m} &= \dots ab^m(ab^{m-1})^{m-1}ab^m(ab^{m-1})^m\dots.
\end{aligned}$$

Thus, we define $H_{0,m}$ to be the sequence obtaining by concatenating all of these strings. That is, $H_{0,m}$ would look as follows:

$$H_{0,m} = \dots ab^m(ab^{m-1})^{m-1}ab^m(ab^{m-1})^m\dots.$$

$\square$

Lemma 5.3. We have $\mathscr{F}_m^n(H_0) = H_0$ for all $n \geq 1$.

Proof. By induction, it suffices to show that $\mathscr{F}_m(H_0) = H_0$. Our strategy will be to keep track of *center* $|$, i.e, a position that will allow us to index our happy sequence as $\dots, -2, -1, 0, 1, 2, \dots$. We will keep track of the center by noting that applying $\mathscr{F}_m$ does not change our previous choice for $|$. The process is below:

$$\begin{aligned}
ba &= b|a \\
\mathscr{F}_m(ab) &= ab^{m-1}|ab^m \\
&= a(b\dots b)|a(bb\dots b) \\
\mathscr{F}_m^2(ab) &= ab^m(ab^{m-1})^{m-1}|ab^m(ab^{m-1})^m \\
&= ab^m(ab^{m-1}\dots ab^{m-1})|ab^m(ab^{m-1}ab^{m-1}\dots ab^{m-1}) \\
&\vdots \\
H_{0,m} &= \dots ab^m(ab^{m-1})^{m-1}|ab^m(ab^{m-1})^m\dots.
\end{aligned}$$

Because of Lemma 2.1, $\mathscr{F}_m^{n-1}(ba) \subset \mathscr{F}_m^n(ba)$, so in particular, the same center that we chose for $\mathscr{F}_m^{n-1}(ba)$ also works for $\mathscr{F}_m^n(ba)$. That is, the center $|$ does not change when we apply $\mathscr{F}_m$. With this center as a reference, if we enumerate letters to the left as $0, -1, -2, \ldots$ and letters to the right as $1, 2, 3, \ldots$, it is clear that the n-th letter of H_0 is the same letter as the n-th letter of $\mathscr{F}_m(H_0)$. Hence, $\mathscr{F}_m(H_0) = H_0$, as desired. □

Theorem 5.1. There exists at least one happy sequence; in particular, H_0 is a happy sequence.

Proof. It suffices to show that H_0 is happy. However, by Lemma 2.3, given any m, we have that

$$H_0 = \mathscr{F}_m^n(H_0),$$

so H_0 is in the image of $\mathscr{F}^n$ for all n, and is thus is happy, as desired. □

Now that we have constructed an m-happy sequence, $H_{0,m}$, we will calculate the density of a's and b's in this sequence as a function of m. Clearly, the rate of growth of the number of bs in a sequence as $\mathscr{F}_m$ is applied depends on m. The greater m is, the more bs get added with every application of $\mathscr{F}_m$. This means the density of a's decreases with respect to an increase in m. We can compute this density of a's explicitly for $H_{0,m}$ using the same matrix method as before.

Proposition 5.1. Consider $\mathscr{F}^\infty(s) \in \mathcal{H}_{0,m}$. We have that $\rho_a(\mathscr{F}^\infty(s)) = \frac{\sqrt{m^2+4}-m}{2}$ and $\rho_b(\mathscr{F}^\infty(s)) = \frac{m+2-\sqrt{m^2+4}}{2}$.

Proof. Let a_n and b_n be the number of a's and b's of $\mathscr{F}_m^n(s)$. Since $a \mapsto ab^m$ and $b \mapsto ab^{m-1}$, we have that

$$a_{n+1} = a_n + b_n \text{ and } b_{n+1} = ma_n + (m-1)b_n$$

Writing these identities in matrix form, we obtain

$$\begin{pmatrix} 1 & 1 \\ m & m-1 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \\ m & m-1 \end{pmatrix}^n \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

so we can diagonalize the recurrence matrix to get

$$A \begin{pmatrix} \left(\frac{m}{2} - \frac{\sqrt{m^2+4}}{2}\right)^n & 0 \\ 0 & \left(\frac{m}{2} + \frac{\sqrt{m^2+4}}{2}\right)^n \end{pmatrix} A^{-1} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

where

$$A = \begin{pmatrix} \frac{2}{m-\sqrt{m^2+4}-2} & \frac{2}{m+\sqrt{m^2+4}-2} \\ 1 & 1 \end{pmatrix}.$$

In the case of the happy sequence H_0, $a_0 = b_0 = 1$. Computing the limit as in section 4, we get that $\rho_a(\mathscr{F}^\infty(s)) = \frac{\sqrt{m^2+4}-m}{2}$, as desired. $\square$

6 Individual Contributions

We refer to the authors Julianne Flusche, Luis Modes, and Daniel Santiago as authors J, L, and D, respectively. Author J wrote sections 1 and 4 and helped to write section 5. Author L wrote section 2 and helped to write sections 3 and 5. Author D wrote sections 3 and 6 and helped to write section 2. All authors contributed to editing and proofreading all sections of the paper.